Math 42-Number Theory
Problem Set #8
Due Tuesday, April 19, 2011

5. Prove that if p and q are distinct odd primes, then

$$\sum_{k=1}^{\frac{p-1}{2}} \left\lfloor \frac{kq}{p} \right\rfloor + \sum_{\ell=1}^{\frac{q-1}{2}} \left\lfloor \frac{\ell p}{q} \right\rfloor = \frac{p-1}{2} \cdot \frac{q-1}{2}.$$

Solution: Consider the rectangle with corners $(0,0)$, $(p,0)$, $(0,q)$ and (p,q). Draw the diagonal from $(0,0)$ to (p,q), the vertical line $x = p/2$, and the horizontal line $y = q/2$. Notice that since p and q are distinct odd primes, there are no lattice points on any of these lines. By counting the lattice points in the rectangle with corners $(0,0)$, $(p/2,0)$, $(0,q/2)$ and $(p/2,q/2)$, we will prove the statement.

On the one hand, the lattice points in this rectangle form an array with $(p-1)/2$ columns and $(q-1)/2$ rows. Therefore, there are $\frac{p-1}{2} \cdot \frac{q-1}{2}$ lattice points in the rectangle.

On the other hand, we can count the lattice points in the lower triangle and in the upper triangle. In the lower triangle, we will count lattice points with first coordinate k. The line $x = k$ intersects the diagonal from $(0,0)$ to (p,q) at $(k, qk/p)$. Therefore, there are $\lfloor qk/p \rfloor$ lattice points in this column. Summing over the possible k's, we get that there are $\sum_{k=1}^{(p-1)/2} \lfloor qk/p \rfloor$ lattice points in the lower triangle.

Similarly, by counting lattice points with second coordinate ℓ in the upper triangle, we get that the upper triangle has $\sum_{\ell=1}^{(q-1)/2} \lfloor p\ell/q \rfloor$ lattice points in it. Since there are no lattice points on the boundaries, we get that

$$\sum_{k=1}^{\frac{p-1}{2}} \left\lfloor \frac{kq}{p} \right\rfloor + \sum_{\ell=1}^{\frac{q-1}{2}} \left\lfloor \frac{\ell p}{q} \right\rfloor = \frac{p-1}{2} \cdot \frac{q-1}{2}.$$